Dept. of Math./CMA University of Oslo
Pure Mathematics No 26
ISSN 0806–2439 December 2010

Kyle-Back's model with Lévy noise *

José Manuel Corcuera†, Gergely Farkas ‡,
Giulia di Nunno§, Bernt Øksendal¶

December 24, 2010

Abstract

The continuous-time version of Kyle's [6] model, known as the Back's [2] model, of asset pricing with asymmetric information, is studied. A larger class of price processes and a larger classes of noise traders' processes are studied. The price process, as in Kyle's [6] model, is allowed to depend on the path of the market order. The process of the noise traders' is considered to be an inhomogeneous Lévy process. The solutions are found with the use of the Hamilton-Jacobi-Bellman equations. With the informed agent being risk-neutral, the price pressure is constant over time, and there is no equilibirium in the presence of jumps. If the informed agent is risk-averse, there is no equilibirium in the presence of either jumps or drift in the process of the noise traders'.

Key words: Market microstructure, insider trading, stochastic control, Lévy processes, semimartingales.
JEL-Classification C61· D43· D44· D53· G11· G12· G14

1 Introduction

Models of markets with the presence of an insider, that is to say, a trader who has some kind of additional information, have a great literature. In the approaches, we can distinguish two fundamentally different ones. One approach is considering the market with a bond and some stocks with prices given exogenously by their dynamics. The other one follows the idea of Kyle [6] where

*The work of J. M. Corcuera is supported by the NILS Grant.

†Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, E-08007 Barcelona, Spain. E-mail: jmcorcuera@ub.edu

‡Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, E-08007 Barcelona, Spain. E-mail:farkasge@gmail.com

§University of Oslo, Centre of Mathematics for Applications, P.O. Box 1053 Blindern NO-0316 Oslo, Norway. E-mail: g.d.nunno@cma.uio.no

¶University of Oslo, Centre of Mathematics for Applications, P.O. Box 1053 Blindern NO-0316 Oslo, Norway. The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087]. E-mail: bernt.oksendal@cma.uio.no

the price of the risky asset is led by the demand of the informed trader through some pricing rule. In the second case, the aim is to find or characterize an equilibrium where the informed agent maximizes his profits and the prices are set in a competitive way. In between one can find the model described by Lassere [7], where a bond and two risky assets are considered, one risky asset with prices given exogenously and one priced as it is in Kyle [6] (and Back [2]). A more general model is studied in Lassere [8], where more risky assets are involved. Following the Kyle-Back approach, Campi and Çetin [4] find equilibrium in the market of zero coupon bonds with default, and so does Back [3] in a market with options. Also the present paper follows the Kyle-Back approach but considers a time continuous trading where the noise traders' dynamics are allowed to have jumps. We study the existence of equilibria in this market model in presence of an insider taking advantage of asymmetric information, and we also consider different types of insider attitude to risk: both risk neutral and risk-adverse.

The paper is organized as follows. In the next Section, the model is described and we formulate the wealth process. In Section 3, one can find an analysis of equilibrium and of its existence, and in the last Section a conclusion is contained.

2 The Model

We consider a market with two assets: we have a risky asset S and a bank account with interest rate r equal to zero for the sake of simplicity. The period in which the participants trade is $[0,1]$. There is to be a public release of information at time 1. The announcement reveals the value of the risky asset, at which price it will trade afterwards (that is to say, at time 1+). This value is denoted by V and it is assumed to be a random variable with finite expectation The market is continuous in time and order driven. There are three kinds of traders. Noise or liquidity traders, who trade for liquidity or hedging reasons, the informed trader or insider, who is aware of the privilege information at time 0, and market makers, who set the price and clear the market. All random variables are defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Denote the price of the stock at time t by P_t and $\mathbb{F}^P = \left(\mathcal{F}_t^P\right)_{0\le t\le 1}$ where $\mathcal{F}_t^P = \sigma(P_s, 0 \le s \le t)$. Let Z be the aggregate demand process of the noise traders. The model we consider is an extension of the one in Back [2], where Z is a Brownian motion with a fixed volatility, to more general set of processes. In Aase *et al.* [1] the authors consider a noise trader's demand with time-varying volatility. In this paper we consider processes that may have a drift and jumps, as well. More precisely we assume that

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t + \mathrm{d}L_t, t \in [0,1], Z_0 = 0. \tag{1}$$

where B is a Brownian motion, independent of V, and $\mu, \sigma : [0,1] \to \mathbb{R}$ are deterministic, càdlàg functions, and L is an pure jump Lévy process independent of V and B. We also assume that the process L can be expressed by

$$L_t = \int_0^t \int_{\mathbb{R}} x\tilde{M}(\mathrm{d}t, \mathrm{d}x),$$

where $\tilde{M}(\mathrm{d}t, \mathrm{d}x) = M(\mathrm{d}t, \mathrm{d}x) - v_t(\mathrm{d}x)\mathrm{d}t$ is the compensated Poisson random measure associated with L, and with intensity $v_t(\mathrm{d}x)$.

Let X be the demand process of the informed trader. At time t, he knows V, as well as $\{P_s : 0 \leq s \leq t\}$, thus, X has to be adapted to the augmented filtration (completed with $\mathbb{P}$-null sets)

$$\mathbb{F}^{V,P} := \left(\mathcal{F}_t^{V,P}\right)_{0 \leq t \leq 1},$$

where

$$\mathbb{F}_t^{V,P} := \sigma(V, P_s, 0 \leq s \leq t)$$

generated by the random variable V and the process P. Because of the independency assumed before, B is an $\mathbb{F}^{V,Z}$-Brownian motion and L is an $\mathbb{F}^{V,Z}$-pure jump Lévy process as well. The informed trader tries to maximize his final wealth, that is, he is risk-neutral (one may find a model with risk averse informed traders in Cho [5] and we also study them in Subsection 3.5). Finally, the market makers "clear" the market by fixing a competitive or rational price, given by

$$P_t = \mathbb{E}(V | Y_s, 0 \leq s \leq t), t \in [0, 1]$$

where $Y = X + Z$ is the total demand that market makers observe. Note that (P_t) is an $\mathbb{F}^Y$-martingale, where $\mathbb{F}^Y = \left(\mathcal{F}_t^Y\right)_{0 \leq t \leq 1}$ and $\mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t)$. Here and in the sequel we always consider $\mathbb{P}$-augmented filtrations.

2.1 The wealth process

In the original model of Kyle, the current price depends on the past demand, while in Back's one it is supposed to be Markovian, depending only on the current total demand. Cho [5] shows that Back's results hold in the original settings with the current price depending on the whole path. We also consider this case. Suppose that the market makers fix prices through a pricing rule, in terms of formulas,

$$P_t = H(t, \xi_t), t \in [0, 1]$$

with

$$\xi_t := \int_0^t \lambda(s) \mathrm{d}Y_s$$

where, the so-called price pressure, λ is a positive smooth function, $H \in C^{1,2}$ and $H(t, \cdot)$ is strictly increasing for every $t \in [0, 1]$. We also write $\xi(t, Y_t)$ for ξ_t. Note then that $\mathbb{F}^Y = \mathbb{F}^P$ and that $\mathbb{F}^{V,P} = \mathbb{F}^{V,Y} = \mathbb{F}^{V,X+Z}$. So it is natural, in this context, to assume that X is adapted to the filtration $\mathbb{F}^{V,Z}$, and that consequently $\mathbb{F}^Y \subseteq \mathbb{F}^{V,Z}$, in such a way that if $X_t = f(Y_s, 0 \leq s \leq t, V)$ for certain measurable function f we can write $X_t = g(Z_s, 0 \leq s \leq t, V)$ for another measurable function g.

Definition 1 *Denote the class of such pairs (H, λ) above by $\mathcal{H}$. An element of $\mathcal{H}$ is called a pricing rule.*

As shown in Back [2] and Cho [5], in equilibrium, the optimal strategies are of the form

$$\mathrm{d}X_t = \theta_t \mathrm{d}t. \tag{2}$$

Definition 2 *Denote, by $\mathcal{X}$, the set of $\mathcal{F}^{V,Z}$-adapted processes X satisfying (2) and such that $\forall\,(H,\lambda) \in \mathcal{H}$*

$$E\left(\int_0^1 \sigma_t^2 U\left(t, \int_0^t \lambda_s \mathrm{d}\left(X_s + Z_s\right)\right)^2\right) dt < \infty \tag{3}$$

$$\int_0^1 \int_{\mathbf{R}} u^2 E\left(U\left(t, \int_0^t \lambda_s \mathrm{d}\left(X_s + Z_s\right) + \lambda_t u\right)^2\right) \nu_t(du) dt < \infty \tag{4}$$

for both cases $U = H$ and $U = \frac{\partial}{\partial y} H$. The elements of $\mathcal{X}$ are called the strategies. We assume that $X \equiv 0$ is a strategy in $\mathcal{X}$.

Later, in Subsection 3.2, we will see that this class can be extended to the one considered in Back [2].

The final wealth W of the insider, just after the announcement, is computed as follows. Consider first a discrete model where trades are made at times $i = 1, 2, \ldots N$. If at time $i-1$, there is an order of buying $X_i - X_{i-1}$ shares, its cost will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by

$$-P_i(X_i - X_{i-1}).$$

Then the total change is

$$-\sum_{i=1}^{N} P_i(X_i - X_{i-1}),$$

and due to the announcement, just after the final time N, by the liquidation of the assets, there is the extra income: $X_N V$. So, the total wealth generated is

$$\begin{aligned} W_{N^+} &= -\sum_{i=1}^{N} P_i(X_i - X_{i-1}) + X_N V \\ &= -\sum_{i=1}^{N} P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{N} (P_i - P_{i-1})(X_i - X_{i-1}) + X_N V \\ &= \sum_{i=1}^{N} (V - P_{i-1})(X_i - X_{i-1}) - \sum_{i=1}^{N} (P_i - P_{i-1})(X_i - X_{i-1}), \end{aligned}$$

where, without loss of generality, we assume $X_0 = 0$. Analogously, in the continuous model,

$$W_{1+} = \int_0^1 (V - P_{t-}) \mathrm{d}X_t - [P, X]_1, \tag{5}$$

where (and throughout the whole article) X_{t-} denotes the left limit $\lim_{s\uparrow t} X_s$. We require that X is an $\mathbb{F}^{V,P}$-semimartingale, so that the integral can be seen as an Itô integral, and to ensure the quadratic covariation $[P, X]$ is finite we also assume that P is an $\mathbb{F}^{V,P}$-semimartingale.

As mentioned before, in equilibrium the market makers fix the pricing rule in a rational way and the insider tries to maximize his expected profit. Formally,

Definition 3 *Given a trading strategy X (and total demand $Y = X + Z$), the price process P is* rational*, if*

$$P_t = \mathbb{E}(V|Y_s, 0 \leq s \leq t), t \in [0, 1]$$

Definition 4 *A strategy X is called* optimal *with respect to a price process P if it maximizes $E(W_{1+})$.*

And if both hold, we have an equilibrium. We are looking for an equilibrium only in the class of pricing rules satisfying Definition 1.

Definition 5 *Let $(H, \lambda) \in \mathcal{H}$ and $X \in \mathcal{X}$. The triple (H, λ, X) is an equilibrium, if the price process $P. := H(\cdot, \xi(\cdot, Y))$ is rational, given X, and the strategy X is optimal, given P.*

3 Equilibrium

As done in Back [2] and Cho [5], we look for an equilibrium and characterize it by using the Hamilton-Jacobi-Bellman equation, as follows. First, we find the equation corresponding to our problem and give a solution to it. Then, we show that there is no loss of generality by assuming (2), and present some properties of rational pricing rules. Finally, we show that when considering the noise traders' demand process (1), there is no equilibrium in the presence of jumps. Moreover if we consider a risk-averse informed trader we may find an equilibrium *only* if there is neither drift, nor jump part in the noise traders' process, thus it leads back to the problem and solution one can found in Cho [5].

3.1 Hamilton-Jacobi-Bellman Equation

Let W be the portfolio wealth of the insider, by (2) and (5)

$$W_{1+} = \int_0^1 (V - P_t)\theta_t \mathrm{d}t.$$

Define the conditional value function as

$$J(V, t, y) := \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\int_t^1 (V - P_l)\tilde{\theta}_l \mathrm{d}l \middle| \mathcal{F}_t^{Z,V} \right],$$

where $\tilde{\theta}_l$ is $\mathcal{F}_l^{P,V}$-measurable, note that we assume that $\mathbb{E}\left[\int_t^1 (V - P_l)\tilde{\theta}_l \mathrm{d}l \middle| \mathcal{F}_t^{Z,V}\right]$ is a measurable function of $\xi(t,\tilde{\theta}) := \int_0^t \lambda_l dY_l^{\tilde{\theta}}$, where $Y_t^{\tilde{\theta}} = Z_t + \int_0^t \tilde{\theta}_l \mathrm{d}l$. According with our framework we work with a pricing rule giving rational prices, i.e., $P_l = \mathbb{E}\left[V \left| Y_s, 0 \le s \le l\right.\right] = H(l, \xi(l,\tilde{\theta}))$. So, the conditional value function can be written as

$$J(V,t,y) = \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\int_t^1 (V - H(l,\xi(l,\tilde{\theta}))\tilde{\theta}_l \mathrm{d}l \middle| \mathcal{F}_t^{Z,V}\right].$$

The expected final wealth is $J(V,0,0)$.

Theorem 6 *Consider an equilibrium with strategy $X \in \mathcal{X}$ for some $\mathbb{F}^{Z,P}$-measurable process θ, and the pricing rule $(H,\lambda) \in \mathcal{H}$. If $J(V,t,y) \equiv J(t,y)$ is smooth then it is a solution of*

$$\lambda_t \frac{\partial J}{\partial y}(t,y) = H(t,y) - V \quad \forall (t,y) \in (0,1] \times \mathbb{R}, \tag{6}$$

and, for all $(t,y) \in (0,1) \times \mathbb{R}$, we have

$$\begin{aligned} 0 &= \frac{\partial J}{\partial t} + \lambda_t \mu_t \frac{\partial J}{\partial y} + \frac{1}{2}\lambda_t^2 \sigma_t^2 \frac{\partial^2 J}{\partial y^2} \\ &+ \int_{\mathbb{R}} \left(J(t, y + \lambda_t u) - J(t,y) - u\lambda_t \frac{\partial J}{\partial y}(t,y)\right) \nu_t(\mathrm{d}u), \end{aligned} \tag{7}$$

By $J(V,t,y)$ being smooth we understand that it has to be continuously differentiable in the second variable t on $(0,1)$ and twice continuously differentiable in the third variable y on $\mathbb{R}$.

Proof. We have that

$$J(t,y) = \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\int_t^1 (V - H(l,\xi(l,\tilde{\theta}))\tilde{\theta}_l \mathrm{d}l \middle| \mathcal{F}_t^{Z,V}\right].$$

Then, by splitting the integral at $t+h \in (t,1)$ we get

$$\begin{aligned} J(t,y) &= \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\int_t^{t+h} (V - H(l,\xi(l,\tilde{\theta})))\tilde{\theta}_l \mathrm{d}l \right. \\ &\qquad \left. + \int_{t+h}^1 (V - H(l,\xi(l,\tilde{\theta})))\tilde{\theta}_l \mathrm{d}l \middle| \mathcal{F}_t^{Z,V}\right]. \end{aligned}$$

Now

$$\begin{aligned} J(t,y) &= \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\int_t^{t+h} (V - H(l,\xi(l,\tilde{\theta})))\tilde{\theta}_l \mathrm{d}l \right. \\ &\quad \left. + \sup_{\hat{\theta}:\xi(t+h,\hat{\theta})=\xi(t+h,\tilde{\theta})} \mathbb{E}\left(\int_{t+h}^1 (V - H(l,\xi(l,\hat{\theta})))\hat{\theta}_l \mathrm{d}l \middle| \mathcal{F}_{t+h}^{Z,V}\right) \middle| \mathcal{F}_t^{Z,V}\right] \end{aligned}$$

hence, we can substitute the second term by $J(V, t+h, \xi(t+h, \tilde{\theta}))$ and we have

$$J(t,y) = \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\int_t^{t+h} (V - H(l, \xi(l,\tilde{\theta})))\tilde{\theta}_l \mathrm{d}l + J(t+h, \xi(t+h,\tilde{\theta})) \middle| \mathcal{F}_t^{Z,V} \right].$$

By subtracting the left hand side of the equation from both sides, we obtain the following expression under expectation operator

$$\int_t^{t+h} \left(V - H(l, \xi(l,\tilde{\theta})) \right) \tilde{\theta}_l \mathrm{d}l + J(t+h, \xi(t+h,\tilde{\theta})) - J(t, \xi(t,\tilde{\theta})).$$

Since $\mathrm{d}X_t = \tilde{\theta}_t \mathrm{d}t$, and

$$\mathrm{d}\xi(t,\tilde{\theta}) = \lambda_t \mathrm{d}Y_t = \lambda_t(\tilde{\theta}_t \mathrm{d}t + \mathrm{d}Z_t) = \lambda_t \tilde{\theta}_t \mathrm{d}t + \lambda_t \mu_t \mathrm{d}t + \lambda_t \sigma_t \mathrm{d}B_t + \lambda_t \mathrm{d}L_t, \quad (8)$$

by the smoothness of J, Itô's formula for J in $\xi(t,\tilde{\theta})$ says

$$\begin{aligned} J(t+h, \xi(t+h,\tilde{\theta})) &= J(t, \xi(t,\tilde{\theta})) \\ &+ \int_t^{t+h} \left[\frac{\partial J}{\partial t}(s, \xi(s,\tilde{\theta})) + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 J}{\partial y^2}(s, \xi(s,\tilde{\theta})) \right] \mathrm{d}s \\ &+ \int_t^{t+h} \frac{\partial J}{\partial y}(s, \xi(s-,\tilde{\theta})) \mathrm{d}\xi(s,\tilde{\theta}) \\ &+ \sum_{t \le s \le t+h} \left[\Delta J(s, \xi(s,\tilde{\theta})) - \frac{\partial J}{\partial y}(s, \xi(s-,\tilde{\theta})) \Delta\xi(s,\tilde{\theta}) \right], \end{aligned}$$

Then, taking into account (8), we obtain

$$\begin{aligned} J(t+h, \xi(t+h,\tilde{\theta})) &= J(t, \xi(t,\tilde{\theta})) \\ &+ \int_t^{t+h} \left[\frac{\partial J}{\partial t} + \lambda_s(\mu_s + \theta_s)\frac{\partial J}{\partial y} + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 J}{\partial y^2} \right] \mathrm{d}s \\ &+ \int_t^{t+h} \lambda_s \sigma_s \frac{\partial J}{\partial y} \mathrm{d}B_s + \int_t^{t+h} \lambda_s \frac{\partial J}{\partial y} \mathrm{d}L_s \\ &+ \sum_{t \le s \le t+h} \left[\Delta J(s, \xi(s,\tilde{\theta})) - \frac{\partial J}{\partial y} \Delta\xi(s,\tilde{\theta}) \right]. \end{aligned}$$

Since $\Delta\xi(t,\tilde{\theta}) = \lambda_s \Delta Y_s = \lambda_s \Delta Z_s$, we have

$$\begin{aligned} & E\left[\sum_{t \le s \le t+h} \Delta J(s, \xi(s,\tilde{\theta})) - \frac{\partial J}{\partial y} \Delta\xi(s,\tilde{\theta}) \middle| \mathcal{F}_t^{P,V} \right] \\ = \; & E\left[\sum_{t \le s \le t+h} J\left(s, \xi(s-,\tilde{\theta}) + \lambda_s \Delta Z_s\right) - J\left(s, \xi(s-,\tilde{\theta})\right) - \lambda_s \Delta Z_s \middle| \mathcal{F}_t^{P,V} \right] \\ = \; & \int_t^{t+h} \int_{\mathbb{R}} E\left[J(s, \xi_{s-} + \lambda_s u) - J(s, \xi_{s-}) - u\lambda_s \frac{\partial J}{\partial y} \middle| \mathcal{F}_t^{P,V} \right] \nu_s(\mathrm{d}u)\mathrm{d}s. \end{aligned}$$

Therefore, we obtain the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} 0 &= \sup_{\theta}\Big\{(V-H)\theta_t+\frac{\partial J}{\partial t}+\lambda_t\theta_t\frac{\partial J}{\partial y}+\lambda_t\mu_t\frac{\partial J}{\partial y}+\frac{1}{2}\lambda_t^2\sigma_t^2\frac{\partial^2 J}{\partial y^2} \\ &\quad +\int_{\mathbb{R}}(J(t,y+\lambda_t u)-J(t,y)-u\lambda_t\frac{\partial J}{\partial y}(t,y))\nu_t(\mathrm{d}u)\Big\} \end{aligned}$$

Note, that since the HJB Equation is linear in θ, its coefficient has to equal 0, otherwise there cannot be a finite maximum. Thus, we obtain (7) and (6) where the $t=1$ case follows from the continuity of $\frac{\partial J}{\partial y}$ and H. ■

The following lemma will play an important role later.

Lemma 7 *Assume that a process G is $\mathbb{F}^Y$-adapted and*

$$G_t = M_t + \int_0^t \alpha_s ds,$$

where M is an $\mathbb{F}^{Z,V}$-martingale and α is $\mathbb{F}^{Z,V}$-adapted and such that for all $t \geq 0$, $\int_0^t \mathbb{E}(|\alpha_s|)\,\mathrm{d}s < \infty$. Let $\mathbb{H}$ be a filtration such that $\mathbb{F}^Y \subseteq \mathbb{H} \subseteq \mathbb{F}^{Z,V}$. Then

$$G_t = N_t + \int_0^t \mathbb{E}\left[\alpha_s \left|\mathcal{H}_s\right.\right] ds,$$

where N is an $\mathbb{H}$-martingale.

Proof. First, we show that $\mathbb{E}\left[M_t\left|\mathcal{H}_t\right.\right]$ is an $\mathbb{H}$-martingale. Let $s \leq t \leq 1$, then since $\mathcal{H}_s \subseteq \mathcal{F}_s^{Z,V}$

$$\mathbb{E}\left[\mathbb{E}\left[M_t\left|\mathcal{H}_t\right.\right]\right|\mathcal{H}_s] = \mathbb{E}\left[M_t\left|\mathcal{H}_s\right.\right] = \mathbb{E}\left[\mathbb{E}\left[M_t\left|\mathcal{F}_s^{Z,V}\right.\right]\middle|\mathcal{H}_s\right] = \mathbb{E}\left[M_s\left|\mathcal{H}_s\right.\right],$$

since M is an $\mathbb{F}^{P,V}$-martingale. Then, consider

$$G_t - G_s = M_t - M_s + \int_s^t \alpha_u \mathrm{d}u.$$

We have

$$\begin{aligned} \mathbb{E}\left[G_t-G_s\right|\mathcal{H}_s] &= \mathbb{E}\left[M_t-M_s\right|\mathcal{H}_s]+\int_s^t\mathbb{E}\left[\alpha_u\right|\mathcal{H}_s]\,\mathrm{d}u \\ &= \mathbb{E}\left[\int_s^t\mathbb{E}\left[\alpha_u\right|\mathcal{H}_u]\,\mathrm{d}u\middle|\mathcal{H}_s\right], \end{aligned}$$

so

$$\mathbb{E}\left[G_t-G_s-\int_s^t\mathbb{E}\left[\alpha_u\right|\mathcal{H}_u]\,\mathrm{d}u\middle|\mathcal{H}_s\right]=0,$$

hence, $N_t := G_t - \int_0^t \mathbb{E}\left[\alpha_u\right|\mathcal{H}_u]\,\mathrm{d}u$ is an $\mathbb{H}$-martingale. ■

Proposition 8 *Let* (H,λ) *be a pricing rule of class* $\mathcal{H}$ *that satisfies*

$$\begin{aligned} 0 &= \frac{\partial H}{\partial t} + \lambda_t \mu_t \frac{\partial H}{\partial y} + \frac{1}{2}\lambda_t^2 \sigma_t^2 \frac{\partial^2 H}{\partial y^2} \\ &+ \int_{\mathbb{R}} \left(H\left(t, y + \lambda_t u\right) - H\left(t,y\right) - u \lambda_t \frac{\partial H}{\partial y}\left(t,y\right) \right) \nu_t\left(\mathrm{d}u\right). \end{aligned} \tag{9}$$

and $X = \int_0^{\cdot} \theta_s \mathrm{d}s$ *a strategy in* $\mathcal{X}$ *. Then the following conditions are equivalent:*

i) The process $(H(t,\xi_t))$ *is an* $\mathbb{F}^Y$*-martingale.*
ii) $\mathbb{E}\left[\theta_t \middle| \mathcal{F}_t^Y\right] = 0$, *and*
iii) The process $\left(Y_t - \int_0^t \mu_s \mathrm{d}s\right)$ *is an* $\mathbb{F}^Y$*-martingale.*

Proof. Let (H,λ) be a pricing rule, then Itô's formula says

$$\begin{aligned} H(t,\xi_t) &= H\left(0,0\right) + \int_0^t \lambda_s \theta_s \frac{\partial H}{\partial y}\left(s,\xi_s\right) \mathrm{d}s \\ &+ \int_0^t \left[\frac{\partial H}{\partial t}\left(s,\xi_s\right) + \frac{\partial H}{\partial y}\left(s,\xi_s\right) \lambda_s \mu_s + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 H}{\partial y^2}(s,\xi_s) \right] \mathrm{d}s \\ &+ \int_0^t \frac{\partial H}{\partial y}(s,\xi_{s-}) \left(\lambda_s \sigma_s \mathrm{d}B_s + \lambda_s \mathrm{d}L_s\right) \\ &+ \sum_{0 \le s \le t} \left[\Delta H(s,\xi_s) - \frac{\partial H}{\partial y}(s,\xi_{s-}) \Delta \xi_s \right] \\ &= M_t + \int_0^t \left[\frac{\partial H}{\partial t}\left(s,\xi_s\right) + \lambda_s \mu_s \frac{\partial H}{\partial y}\left(s,\xi_s\right) + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 H}{\partial y^2}(s,\xi_s) \right] \mathrm{d}s \\ &+ \int_0^t (H(s,\xi_{s-} + \lambda_s u) - H(s,\xi_{s-}) - u\lambda_s \frac{\partial H}{\partial y}(s,\xi_{s-}))\nu_s(\mathrm{d}u)\mathrm{d}s \\ &+ \int_0^t \lambda_s \theta_s \frac{\partial H}{\partial y}\left(s,\xi_s\right) \mathrm{d}s. \end{aligned}$$

where M is an $\mathcal{F}^{Z,V}$-martingale. Then, by Lemma 7 we know that H can be rewritten as

$$\begin{aligned} H(t,\xi_t) &= N_t + \int_0^t \left[\frac{\partial H}{\partial t}\left(s,\xi_s\right) + \frac{\partial H}{\partial y}\left(s,\xi_s\right) \lambda_s \mu_s + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 H}{\partial y^2}(s,\xi_s) \right] \mathrm{d}s \\ &+ \int_0^t (H(s,\xi_{s-} + \lambda_s u) - H(s,\xi_{s-}) - u\lambda_s \frac{\partial H}{\partial y}(s,\xi_{s-}))\nu_s(\mathrm{d}u)\mathrm{d}s \\ &+ \int_0^t \lambda_s \mathbb{E}(\theta_s | \mathcal{F}_s^Y) \frac{\partial H}{\partial y}\left(s,\xi_s\right) s \\ &= N_t + \int_0^t \lambda_s \mathbb{E}(\theta_s | \mathcal{F}_s^Y) \frac{\partial H}{\partial y}\left(s,\xi_s\right) \mathrm{d}s, \end{aligned}$$

where N is an $\mathbb{F}^Y$-martingale. Then, $(H(t,\xi_t))$ is an $\mathbb{F}^Y$-martingale if and only if

$$\mathbb{E}(\theta_s|\mathcal{F}_s^Y) = 0,$$

which proves that i) and ii) are equivalent. Also, we know that

$$Y_t = Z_t + \int_0^t \theta_s \mathrm{d}s,$$

so

$$Y_t - \int_0^t \mu_s \mathrm{d}s = R_t + \int_0^t \theta_s \mathrm{d}s,$$

where R is an $\mathbb{F}^{Z,V}$-martingale. Then we can write, by Proposition 7,

$$Y_t - \int_0^t \mu_s \mathrm{d}s = U_t + \int_0^t \mathbb{E}(\theta_s|\mathcal{F}_s^Y)\mathrm{d}s$$

where U is an $\mathbb{F}^Y$-martingale which proves that ii) and iii) are equivalent. ■

In Back [2], it is proved that, in equilibrium, the pricing rule is of the form

$$H(t,\xi) = \mathbb{E}\left[H\left(1,\xi+\xi_1-\xi_t\right)\right].$$

In Cho [5], we find that in equilibrium, the price pressure λ is constant and the pricing rule is of the same form, and gives a solution of the Hamilton-Jacobi-Bellman equations with the same construction as in Back [2]. The following proposition shows that certain properties hold in our case, as well.

Proposition 9 *Suppose that for (H,λ) there exist a smooth solution J such that (H,λ,J) is a solution of (6) and (7), then the price pressure λ_t is constant and the pricing rule is of the form*

$$H(t,y) = \mathbb{E}\left[H(1,y+\lambda(Z_1-Z_t)\right]. \tag{10}$$

Conversely, if the price pressure is constant, and H satisfies (9), then $(H,,J)$ with J defined by

$$\begin{aligned} J\left(t,y\right) &= \mathbb{E}\left[J(1,y+\lambda(Z_1-Z_t))\right], \\ J\left(1,\cdot\right) &= \int_{\cdot}^{H^{-1}(1,\lambda\cdot)(V)} \frac{V-H\left(1,x\right)}{\lambda}\mathrm{d}x \end{aligned}$$

(where the expectation is taken over $Z_1 - Z_t$ and V is regarded as a constant) is a solution of (6), (7) with the boundary condition

$$J(1,H^{-1}\left(1,\lambda\cdot\right)(V)) = 0. \tag{11}$$

Proof. By differentiating first in (6) with respect to t, and then in (7) with respect to y and combining the results, we get the following equation for H

$$\begin{aligned} 0 &= \frac{\partial H}{\partial t} + \lambda_t\mu_t\frac{\partial H}{\partial y} + \frac{1}{2}\lambda_t^2\sigma_t^2\frac{\partial^2 H}{\partial y^2} + (V-H)\frac{\lambda_t'}{\lambda_t} \\ &\quad + \int_{\mathbb{R}}\left(H\left(t,y+\lambda_t u\right) - H\left(t,y\right) - u\lambda_t\frac{\partial H}{\partial y}\left(t,y\right)\right)\nu_t\left(\mathrm{d}u\right). \end{aligned} \tag{12}$$

Then since $H(t,y)$ does not depend on V, we have that $\lambda_t' \equiv 0$. By Itô's formula applied to $H(t,\lambda Z_t)$, we obtain

$$H(t,y) = \mathbb{E}\left[H(1,\lambda Z_1) \middle| \lambda Z_t = y\right].$$

Suppose the price pressure is a constant λ, and (H,λ) satisfies (9) and J is given as assumed above. Then,

$$J_y(t,y) = -\frac{1}{\lambda}\left(V - \mathbb{E}\left[H(1,\lambda Z_1)|\lambda Z_t = y\right]\right) = -\frac{V - H(t,y)}{\lambda},$$

which shows that (H,λ,J) satisfies (6). Equation (7) follows from Feynman-Kac's formula, as used above. The boundary condition (11) is straightforwardly verified. ■

3.2 Optimality in a larger class of strategies

In this subsection, we see in what extent there is loss of generality by considering strategies of the form (2). In fact, we do not need to assume in advance that the trading strategy is of the form (2). However we conclude that the optimal strategies are continuous and with bounded variation. We also see that, provided we have a solution of the (6) and (7), the maximum expected profit, given V, is $J(V,0,0)$. So, the following theorem is also a *verification theorem.*

Theorem 10 *If there exists (H,λ,J) satisfying (6), (7) with $(H,\lambda)\in\mathcal{H}$, then for any solution and any strategy X, semimartingale with respect to $\mathbb{F}^{Z,V}$, the informed trader's maximum expected profit, for fixed V, equals $J(V,0,0)$. Moreover this maximum value can be reached by X if and only if it satisfies the following properties:*

(i) X has continuous paths,

(ii) the Doob's decomposition of X does not have martingale part,

(iii) the strategy drives the price to V, that is $\lim_{t\to 1} P_t = V$.

If at least one of these properties does not hold for X, then it is not optimal.

Proof. Having this $J \geq 0$ solution of (6) and (7) we are trying to maximize the expected final wealth

$$\mathbb{E}\left[\int_0^1 (V - P_{t-})\, dX_t - [P,X]_1\right]. \tag{13}$$

Denote, as before, $P_t = H(t,\xi_t)$, the price set by the market makers at time t, and V the insider's information, and $\xi_t := \int_0^t \lambda_s \mathrm{d}Y_s$. We write $J(V,t,\xi_t) \equiv J(t,\xi_t)$. By using Itô's formula, we have

$$\begin{aligned} J(1,\xi_1) &= J(0,\xi_0) + \int_0^1 \frac{\partial J}{\partial y}(t,\xi_{t-})\mathrm{d}\xi_t + \int_0^1 \frac{\partial J}{\partial t}(t,\xi_{t-})\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^1 \frac{\partial^2 J}{\partial y^2}(t,\xi_{t-})\mathrm{d}[\xi^c,\xi^c]_t + \sum_{0\le t\le 1}\left(\Delta J(t,\xi_t) - \frac{\partial J}{\partial y}(t,\xi_{t-})\Delta\xi_t\right). \end{aligned}$$

By construction, $\xi_0 = 0$, and we have $\mathrm{d}\xi_t = \lambda_t \mathrm{d}Y_t$

$$\mathrm{d}[\xi^c, \xi^c]_t = \lambda_t^2 \mathrm{d}[X^c, X^c]_t + 2\lambda_t^2 \mathrm{d}[X^c, Z^c]_t + \lambda_t^2 \sigma_t^2 \mathrm{d}t,$$

so using (6), (7), we get

$$\begin{aligned}
J(1, \xi_1) &= J(0,0) + \int_0^1 (P_{t-} - V)(\mathrm{d}X_t + \sigma_t \mathrm{d}B_t + \mathrm{d}L_t) \\
&\quad + \frac{1}{2} \int_0^1 \frac{\partial^2 J}{\partial y^2}(t, \xi_{t-}) \lambda_t^2 \mathrm{d}[X^c, X^c]_t \\
&\quad + \int_0^1 \frac{\partial^2 J}{\partial y^2}(t, \xi_{t-}) \lambda_t^2 \mathrm{d}[X^c, Z^c] + \sum_{0 \leq t \leq 1} \left(\Delta J(t, \xi_t) - \frac{\partial J}{\partial y}(t, \xi_{t-}) \Delta \xi_t \right) \\
&\quad - \int_0^1 \int_{\mathbb{R}} (J(t, \xi_{t-} + \lambda_t u) - J(t, \xi_{t-}) - \frac{\partial J}{\partial y} u) \nu_t(\mathrm{d}u) \mathrm{d}t
\end{aligned}$$

Subtracting $[P, X]_1$ from both sides and substituting, we obtain

$$\begin{aligned}
&\int_0^1 (V - P_{t-}) \mathrm{d}X_t - [P, X]_1 - J(0,0) \\
= \quad & -J(1, \xi_1) + \int_0^1 (P_{t-} - V)(\sigma_t \mathrm{d}B_t + \mathrm{d}L_t) \\
& + \frac{1}{2} \int_0^1 \frac{\partial^2 J}{\partial y^2}(t, \xi_{t-}) \lambda_t^2 \mathrm{d}[X^c, X^c]_t + \int_0^1 \frac{\partial^2 J}{\partial y^2}(t, \xi_{t-}) \lambda_t^2 \mathrm{d}[X^c, Z^c]_t \\
& + \sum_{0 \leq t \leq 1} \left(\Delta J(t, \xi_t) - \frac{\partial J}{\partial y} \Delta \xi_t \right) \\
& - \int_0^1 \int_{\mathbb{R}} (J(t, \xi_{t-} + \lambda_t u) - J(t, \xi_{t-}) - u \frac{\partial J}{\partial y}(t, \xi_{t-})) \lambda_t \nu_t(\mathrm{d}u) \mathrm{d}t - [P, X]_1.
\end{aligned}$$

We will show that the expectation of the left hand side is non-positive by evaluating the right hand side. Note that

$$[P, X]_1 \equiv [P^c, X^c]_1 + \sum_{0 \leq t \leq 1} \Delta P_t \Delta X_t.$$

Itô's formula for H shows that the continuous local martingale part of P is $\int \frac{\partial H}{\partial y}(t, \xi_{t-}) \mathrm{d}\xi_t^c$, so by using (6), we obtain

$$\begin{aligned}
[P^c, X^c]_1 &= \left[\int \frac{\partial H}{\partial y}(t, \xi_{t-}) \mathrm{d}\xi_t^c, X^c \right]_1 = \int_0^1 \frac{\partial H}{\partial y}(t, \xi_{t-}) \mathrm{d} \left[\xi^c, X^c \right]_t \\
&= \int_0^1 \frac{\partial^2 J}{\partial y^2}(t, \xi_{t-}) \lambda_t^2 \mathrm{d} \left[X^c, X^c \right]_t + \int_0^1 \frac{\partial^2 J}{\partial y^2}(t, \xi_{t-}) \lambda_t^2 \mathrm{d} \left[X^c, Z^c \right]_t,
\end{aligned}$$

and also

$$\begin{aligned}
\lambda_t \frac{\partial J}{\partial y}(t, \xi_{t-}) \Delta X_t + \Delta P_t \Delta X_t &= (P_{t-} - V) \Delta X_t + \Delta P_t \Delta X_t \\
&= (P_t - V) \Delta X_t = \lambda_t \frac{\partial J}{\partial y}(t, \xi_t) \Delta X_t.
\end{aligned}$$

Substituting them for $[P,X]_t$ in the right hand side of equation, it simplifies to

$$
\begin{aligned}
&-J(1,\xi_1)+\int_0^1 (P_{t-}-V)(\sigma_t \mathrm{d}B_t+\mathrm{d}L_t)-\frac{1}{2}\int_0^1 \frac{\partial^2 J}{\partial y^2}(t,\xi_{t-})\mathrm{d}[X^c,X^c]_t\\
&+\sum_{0\le t\le 1}\left(J(t,\xi_t)-J(t,\xi_{t-}+\lambda_t\Delta Z_t)-\lambda_t\frac{\partial J}{\partial y}(t,\xi_t)\Delta X_t\right)\\
&+\sum_{0\le t\le 1}\left(\Delta J(t,\xi_{t-}+\lambda_t\Delta Z_t)-\lambda_t\frac{\partial J}{\partial y}(t,\xi_{t-})\Delta Z_t\right)\\
&-\int_0^1\int_{\mathbb{R}}(J(t,\xi_{t-}+\lambda_t u)-J(t,\xi_{t-})-\lambda_t\frac{\partial J}{\partial y}u)\nu_t(\mathrm{d}u)\mathrm{d}t.
\end{aligned}
$$

1. By definition
$$J(1,y)=\lim_{t\to 1}J(t,y)\ge 0$$
because $J(\cdot,y)$ is smooth and
$$J(t,y)=\sup_{\tilde\theta:\xi(t,\tilde\theta)=y}\mathbb{E}\left[\int_t^1 (V-P_l)\tilde\theta_l \mathrm{d}l\,\middle|\,\mathcal{F}_t^{P,V}\right]\ge 0$$
so, we have that $-J(1,y)\le 0$, for every y then $J(1,\xi_1)=0$ if and only if
$$\lambda_1\frac{\partial J}{\partial y}(1,\xi_1)=H(1,\xi_1)-V=0.$$

2. By conditions (3) and (4) the processes $\int_0^{\cdot}(P_{t-}-V)(\sigma_t \mathrm{d}B_t+\mathrm{d}L_t)$ and
$$
\begin{aligned}
&+\sum_{0\le t\le \cdot}\left(\Delta J(t,\xi_{t-}+\lambda_t\Delta Z_t)-\frac{\partial J}{\partial y}(t,\xi_{t-})\Delta Z_t\right)\\
&-\int_0^{\cdot}\int_{\mathbb{R}}(J(t,\xi_{t-}+\lambda_t u)-J(t,\xi_{t-})-\frac{\partial J}{\partial y}\lambda_t u)\nu_t(\mathrm{d}u)\mathrm{d}t,
\end{aligned}
$$
are $\mathbb{F}^{P,V}$-martingales, so they vanish when we take expectations.

3. By (6) and H being increasing monotone, we have that J_y is increasing, hence $J_{yy}>0$, and the measure $\mathrm{d}[X^c,X^c]\ge 0$,

4. $J_{yy}>0$ (convexity) implies that
$$J(t,x+h)-J(t,x+h_1)-\frac{\partial J}{\partial y}(t,x+h)(h-h_1)\le 0.$$
So,
$$\sum_{0\le t\le 1}\left(J(t,\xi_{t-}+\lambda_t\Delta Y_t)-J(t,\xi_{t-}+\lambda_t\Delta Z_t)-\frac{\partial J}{\partial y}(t,\xi_t)\lambda_t\Delta X_t\right)\le 0,$$
and has its maximum if and only if $\Delta Y_t=\Delta Z_t$, that is if and only if X is continuous.

■

3.3 Rationality

If (H, λ) is a solution of (6) and (7), then, by applying the Itô formula, we have that $\left(H(t, \int_0^t \lambda_s \mathrm{d}Z_s)\right)$ is a square-integrable martingale. Then, without the presence of the insider, the price process follows a martingale. With his presence we want $\left(H(t, \int_0^t \lambda_s \mathrm{d}Y_s)\right)$ to remain a martingale, since, as we will see, this implies that the pricing rule is rational, that is

$$H(t, \xi_t) = \mathbb{E}\left[V|\mathcal{F}_t^Y\right].$$

In fact, we have the following proposition.

Proposition 11 *Suppose, $H \in \mathcal{H}$ is a solution of (6)and (7) and $X \in \mathcal{X}$ optimal such that $\mathbb{E}\left[\theta_t | \mathcal{F}_t^Y\right] = 0$, then the pricing rule is rational, that is*

$$H(t, \xi_t) = \mathbb{E}[V|\mathcal{F}_t^Y], 0 \leq t \leq 1,$$

and (H, X) is an equilibrium.

Proof. By Proposition 8, $H(t, \xi_t)$ is an $\mathbb{F}^Y$-martingale. Then

$$H(t, \xi_t) = \mathbb{E}(H(1, \xi_1)|\mathcal{F}_t^Y)$$

and since X is optimal, $H(1, \xi_1) = V$. ■

3.4 Existence of equilibrium

From Theorem 10 we have seen that necessary and sufficient conditions to have an equilibrium are:

i) to have a price function $H \in \mathcal{H}$ satisfying the equation (9)

ii) to have a strategy $\int_0^{\cdot} \theta_s \mathrm{d}s \in \mathcal{X}$ satisfying the following conditions:

1. the process $\left(Y_t - \int_0^t \mu_s \mathrm{d}s\right)$ is an $\mathbb{F}^Y$-martingale, where $Y_t = \int_0^t \theta_s \mathrm{d}s + Z_t$ is the total demand.
2. it drives the total demand to the value $R := H^{-1}(1, \lambda\cdot)(V)$, that is $Y_1 = R$.

Theorem 12 *If the demand of the liquidity traders Z has a jump component (i.e. $L \neq 0$), then there is not equilibrium.*

Proof. Let Y be the total demand in an equilibrium, then we have

$$M_t := Y_t - \int_0^t \mu_s \mathrm{d}s = \int_0^t \sigma_s \mathrm{d}B_s + L_t + \int_0^t \theta_s(Y_1; Y_u, 0 \leq u \leq s)\mathrm{d}s, 0 \leq t \leq 1$$

so the r.h.s. is the Doob-Meyer decomposition of the $\mathbb{F}^Y$-martingale M in the filtration $\mathbb{F}^{Y,Y_1}$, since $\int_0^{\cdot} \sigma_s \mathrm{d}B_s + L_\cdot$ is an $\mathbb{F}^{Y,Y_1}$-martingale. Now, we can decompose the martingale M in its continuous and jump components,

$$\begin{aligned} M_t^c &= \int_0^t \sigma_s \mathrm{d}B_s + \Gamma_t, \\ M_t^d &= L_t + \Lambda_t. \end{aligned}$$

These above are the $\mathbb{F}^{Y,Y_1}$-Doob-Meyer decompositions of M^c and M^d respectively, with $\Gamma_t + \Lambda_t = \int_0^t \theta_s(Y_1; Y_u, 0 \leq u \leq s)\mathrm{d}s$. Note that we have

$$M_t^d - L_t = \int_0^t \int_{\mathbb{R}} x\left(\delta(\mathrm{d}s, \mathrm{d}x) - \upsilon_t(\mathrm{d}x)\mathrm{d}s\right) = \Lambda_t,$$

where $\left(\int_0^t \int_{\mathbb{R}} x\delta(\mathrm{d}s, \mathrm{d}x)\right)$ is the $\mathbb{F}^Y$-predictable compensator of the integer random measure in the process M^d. So Λ is $\mathbb{F}^Y$-predictable and does not depend on Y_1. Moreover $M_t^d - L_t$ is an $\mathbb{F}^Y$-martingale and consequently $\Lambda \equiv 0$, a.s..

So, if there is only jump part in the demand of liquidity traders, i.e. $Z \equiv L$ $M_t = Y_t = L_t$ and $R = L_1$ contradicting the hypothesis of independence between L and R. Therefore there is not equilibrium.

If, on the contrary, we have a continuous part in Z then the argument above yields

$$M_t^c = \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \theta_s(Y_1; Y_u, 0 \leq u \leq s)\mathrm{d}s, \tag{14}$$

and

$$M_t^d = L_t.$$

Note that, since B is independent of L, (14) is the Doob-Meyer decomposition of M^c in the filtration $(\sigma(Y_1; Y_u, 0 \leq u \leq s; L_u, 0 \leq u \leq 1))$.

From ii.2 we know that to have optimality we need $M_1^c = R - L_1 - \int_0^1 \mu_s \mathrm{d}s$. So, we need to find the Doob-Meyer decomposition of $M_\cdot^c$ in the insider's filtration which is

$$(\sigma(Y_1, Y_s, 0 \leq s \leq t)) = (\sigma(M_1^c + L_1 - L_t, M_s, 0 \leq s \leq t)).$$

By the Dambis-Dubins-Schwarz theorem (see Revuz and Yor [9], Thm. V.1.6. and Prop.V.1.11), $M_t^c \sim \int_0^t \sigma_s \mathrm{d}\tilde{B}_s$ for certain Brownian motion $\tilde{B}$ and then, by using Lemma (7), we have that, in the filtration $(\sigma(M_1^c; M_u^c, 0 \leq u \leq s; L_u, 0 \leq u \leq 1))$, the Doob-Meyer decomposition is given by

$$M_t^c = \int_0^t \sigma_s \mathrm{d}\hat{B}_s + \int_0^t \frac{M_1^c - M_s^c}{\int_s^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s$$

where $\hat{B}$ is a Brownian motion independent of M_1^c and L. Now, again by Lemma (7), we have that the decomposition in the filtration $(\sigma(M_1^c + L_1 - L_t, M_s, 0 \leq s \leq t))$

is given by

$$M_t^c = \int_0^t \sigma_s \mathrm{d}\bar{B}_s + \int_0^t \frac{E(M_1^c | M_1^c + L_1 - L_s; M_u; 0 \le u \le s) - M_s^c}{\int_s^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s,$$

where $\bar{B}$ is a Brownian motion. But $\bar{B}$ depends on $M_1^c = R - L_1 - \int_0^1 \mu_s \mathrm{d}s$ and then on R, since L_1 is independent of the Brownian part by hypothesis. This contradicts the hypothesis of independence of the noise demand process and the privileged information. In fact, if $\bar{B}$ was independent of M_1^c, the following situation would follows: since, for all $0 \le t \le 1$

$$\int_0^t \sigma_s \mathrm{d}\hat{B}_s + \int_0^t \frac{M_1^c - E(M_1^c | M_1^c + L_1 - L_s; M_u; 0 \le u \le s)}{\int_s^1 \sigma_u^2 \mathrm{d}u} \sigma_s \mathrm{d}s = \int_0^t \sigma_s \mathrm{d}\bar{B}_s,$$

and by the symmetry between $\bar{B}$ and $\hat{B}$, we would have that $\int_0^\cdot \sigma_s \mathrm{d}\hat{B}_s$ and consequently $\int_0^\cdot \sigma_s \mathrm{d}\hat{B}_s - \int_0^\cdot \sigma_s \mathrm{d}\bar{B}_s$ would be $\mathbb{F}^{M^c, M_1^c}$-martingales, thus $M_1^c - E(M_1^c | M_1^c + L_1 - L_s; M_u; 0 \le u \le s) = 0$, a.e.. But this would imply in particular that

$$L_1 = E(L_1 | M_1^c + L_1) = E(L_1 | R)$$

contradicting the hypothesis that L is independent of R.

So, in any case of Z with and without continuous component we obtain that L cannot be independent of R if we want to have rational prices. Hence there is not equilibrium. ■

Proposition 13 *If the demand of the liquidity traders, Z, has not a jump component, then the equilibrium strategy is such that*

$$\theta_t = \frac{Y_1 - Y_t - \int_t^1 \mu_s \mathrm{d}s}{\int_t^1 \sigma_s^2 \mathrm{d}s} \sigma_t^2$$

Proof. *If $\bar{Y}_t := Y_t - \int_0^t \mu_s \mathrm{d}s = \int_0^t \sigma_s \mathrm{d}\tilde{B}_s$, where $\tilde{B}$ is a Brownian motion, then*

$$\bar{Y}_t - \int_0^t \frac{\bar{Y}_1 - \bar{Y}_t}{\int_s^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s, 0 \le t \le 1,$$

is a process identical in law to $\int_0^\cdot \sigma_s \mathrm{d}\tilde{B}_s$ and independent of Y_1. ■

3.5 When the insider is risk averse

In this section we study the case of a risk-averse insider. We restrict ourselves to the case of exponential utility. Assume that the insider wants to maximize $E(u(W_{1+})) = E(\gamma e^{\gamma W_{1+}})$, where $\gamma < 0$. Then the value function is given by

$$J(t, y) := \sup_{\tilde{\theta}: \xi(t, \tilde{\theta}) = y} \mathbb{E}\left[\gamma \exp\left\{ \gamma \int_t^1 (V - P_l) \tilde{\theta}_l \mathrm{d}l \right\} \middle| \mathcal{F}_t^{Z,V} \right],$$

and adding and subtracting $\gamma \exp \gamma \int_{t+h}^{1} (V - P_l)\tilde{\theta}_l \mathrm{d}l$ under the expectation, we have

$$
\begin{aligned}
J(t,y) \quad &= \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\gamma \exp\left\{\gamma \int_t^1 (V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\left(1-\exp\left\{-\gamma\int_t^{t+h}(V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\right)\right. \\
&\quad \left. +\gamma \exp\left\{\gamma \int_{t+h}^1 (V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\right| \mathcal{F}_t^{Z,V}\Bigg], \\
&= \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\gamma \exp\left\{\gamma \int_t^1 (V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\left(1-\exp\left\{-\gamma\int_t^{t+h}(V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\right)\right. \\
&\quad \left. +J(t+h,\xi(t+h,\tilde{\theta}))\right| \mathcal{F}_t^{Z,V}\Bigg].
\end{aligned}
$$

So, as done in the risk-neutral case, subtracting $J(t,y)$, we can apply Itô's formula to the difference $J(t+h,\xi(t+h,\tilde{\theta})) - J(t,\xi(t,\tilde{\theta}))$. Moreover note that, as h tends to zero, the limit of

$$
\frac{\left(1-\exp\left\{-\gamma\int_t^{t+h}(V-P_l)\tilde{\theta}_l dl\right\}\right)}{h}
$$

is $\gamma(V-P_t)\tilde{\theta}_t$. Hence, we get the following HJB equations,where of course $P_t = H(t,\xi_t)$.

$$
\begin{aligned}
0 \quad &= \sup_{\theta}\left\{ J\gamma(V-H)\theta_t + \frac{\partial J}{\partial t} + \lambda_t\theta_t\frac{\partial J}{\partial y} + \frac{\partial J}{\partial y}\lambda_t\mu_t + \frac{1}{2}\lambda_t^2\sigma_t^2\frac{\partial^2 J}{\partial y^2}\right. \\
&\quad \left. + \int_{\mathbb{R}} (J(t,y+\lambda_t u) - J(t,y) - u\lambda_t\frac{\partial J}{\partial y}(t,y))\nu_t(\mathrm{d}u)\right\}.
\end{aligned}
$$

Since the equation is linear in θ, we get the following two equations similar to the risk-neutral case:

$$
\lambda_t\frac{\partial J}{\partial y}(t,y) = J\left(t,y\right)\gamma\left(H(t,y)-V\right) \quad \forall (t,y)\in(0,1]\times\mathbb{R}, \tag{15}
$$

and for all $(t,y)\in(0,1)\times\mathbb{R}$

$$
\begin{aligned}
0 \quad &= \frac{\partial J}{\partial t} + \lambda_t\mu_t\frac{\partial J}{\partial y} + \frac{1}{2}\lambda_t^2\sigma_t^2\frac{\partial^2 J}{\partial y^2} \\
&\quad + \int_{\mathbb{R}}\left(J\left(t,y+\lambda_t u\right) - J\left(t,y\right) - u\lambda_t\frac{\partial J}{\partial y}\left(t,y\right)\right)\nu_t\left(\mathrm{d}u\right).
\end{aligned} \tag{16}
$$

Differentiating (15) by y we have

$$
\frac{\partial^2 J}{\partial y^2} = \frac{1}{\lambda_t^2} J\gamma\left[\lambda_t\frac{\partial H}{\partial y} + (H-V)^2\gamma\right],
$$

which plugged in to (16) implies

$$\begin{aligned}
0 &= \frac{\partial J}{\partial t} + (H-V)\,\gamma J \mu_t + \frac{1}{2} J \gamma \sigma_t^2 \left[\lambda_t \frac{\partial J}{\partial y} + (H-V)^2 \gamma \right] \\
&\quad + \int_{\mathbb{R}} \left(J(t, y + \lambda_t u) - J(t,y) - u \lambda_t \frac{\partial J}{\partial y}(t,y) \right) \nu_t(\mathrm{d}u). \qquad (17)
\end{aligned}$$

Denote $\int_{\mathbb{R}} \left(J(t, y + \lambda_t u) - J(t,y) - u \lambda_t \frac{\partial J}{\partial y}(t,y) \right) \nu_t(\mathrm{d}u)$ by $I(t,y)$. By differentiating the previous equation by y, we get

$$\begin{aligned}
0 &= \frac{\partial J}{\partial t \partial y} + \frac{\partial H}{\partial y} \gamma J \mu_t + \frac{(H-V)^2 \gamma^2 J \mu_t}{\lambda_t} \\
&\quad + \frac{1}{2} \gamma \sigma_t^2 \left\{ \frac{(H-V)\,\gamma J}{\lambda_t} \left[\lambda_t \frac{\partial H}{\partial y} + (H-V)^2 \gamma \right] + J \left[\lambda_t \frac{\partial^2 H}{\partial y^2} + 2(H-V) \frac{\partial H}{\partial y} \gamma \right] \right\} \\
&\quad + I_y(t,y), \qquad (18)
\end{aligned}$$

so

$$\begin{aligned}
\frac{\partial J}{\partial t \partial y} &= -J\gamma\mu_t \left(\frac{\partial H}{\partial y} + \frac{(V-H)^2 \gamma}{\lambda_t} \right) \\
&\quad + J \frac{\gamma \sigma_t^2}{2} \left(3\gamma (V-H) \frac{\partial H}{\partial y} + \frac{\gamma^2}{\lambda_t} (V-H)^3 - \lambda_t \frac{\partial^2 H}{\partial y^2} \right) - I_y(t,y).
\end{aligned}$$

While differentiating (15) by t, we get

$$\lambda_t' \frac{\partial J}{\partial y} + \frac{\partial J}{\partial t \partial y} \lambda_t = \frac{\partial H}{\partial t} \gamma J + (H-V)\,\gamma \frac{\partial J}{\partial t}.$$

Inserting this expression together with (15) into (16), we get

$$\begin{aligned}
\frac{\partial J}{\partial t \partial y} &= J \left[(V-H)^2 \frac{\gamma^2}{\lambda_t} \mu_t + \frac{\gamma^3 \sigma_t^2}{2\lambda_t} (V-H)^3 + \frac{\gamma^2 \sigma_t^2}{2} (V-H) \frac{\partial H}{\partial y} \right. \\
&\quad \left. + \frac{\gamma}{\lambda_t} \frac{\partial H}{\partial t} + \gamma \frac{\lambda_t'}{\lambda_t^2} (V-H) \right] + \frac{\gamma (H-V)}{\lambda_t} I(t,y). \qquad (19)
\end{aligned}$$

Subtracting (19) from (18), we obtain

$$\begin{aligned}
0 &= -J\gamma\mu_t \frac{\partial H}{\partial y} + \frac{\partial H}{\partial t} + \frac{1}{2} \sigma_t^2 \lambda_t^2 \frac{\partial^2 H}{\partial y^2} - \lambda_t (V-H) \left[\left(\frac{1}{\lambda_t} \right)' + \gamma \sigma_t^2 \frac{\partial H}{\partial y} \right] \\
&\quad + \frac{\gamma (H-V)}{\lambda_t} I(t,y) - I_y(t,y).
\end{aligned}$$

Also, (15) implies

$$\frac{\frac{\partial J}{\partial y}}{J} = \frac{(H-V)\,\gamma}{\lambda_t}.$$

Hence we have that

$$
\begin{aligned}
J &= \exp\left\{\frac{\gamma}{\lambda_t}\int_0^y (H-V)\,du\right\} c_2(t) =: H^e(t,y)\,c_2(t) \\
J_y &= \frac{\partial H^e}{\partial y} = H^e\frac{\gamma}{\lambda_t}(H-V)\,c_2(t).
\end{aligned}
$$

and

$$
I(t,y) = c_2(t)\int_{\mathbb{R}} \left(H^e(t,y+\lambda_t u) - H^e(t,y) - u\gamma H^e(H(t,y)-V)\right)\nu_t(\mathrm{d}u).
$$

So,

$$
\begin{aligned}
\frac{\gamma(H(t,y)-V)}{\lambda_t} I(t,y) &= c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[(H(t,y)-V)\,H^e(t,y+\lambda_t u)\right. \\
&\quad -(H(t,y)-V)\,H^e(t,y) \\
&\quad \left.-uH^e\gamma(H(t,y)-V)^2\right]\nu_t(\mathrm{d}u)
\end{aligned}
$$

and

$$
\begin{aligned}
I_y(t,y) &= c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[H^e(t,y+\lambda_t u)(H(t,y+\lambda_t u)-V)\right. \\
&\quad -H^e(t,y)(H(t,y)-V) \\
&\quad \left.-u\gamma H^e(t,y)(H(t,y)-V)^2 + u\lambda_t H^e(t,y)\,H_y(t,y)\right]\nu_t(\mathrm{d}u).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\gamma(H-V)}{\lambda_t} I(t,y) - I_y(t,y) &= -c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[H^e(t,y+\lambda_t u)(H(t,y+\lambda_t u)-H(t,y))\right. \\
&\quad \left.-u\lambda_t H^e(t,y)\frac{\partial H}{\partial y}(t,y)\right]\nu_t(\mathrm{d}u).
\end{aligned}
$$

Hence, we get the following equation for H. If there is solution (J,H,λ) satisfying the HJB Equations, (H,λ) has to satisfy

$$
\begin{aligned}
0 &= -H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y} + \frac{\partial H}{\partial t} + \frac{1}{2}\sigma_t^2\lambda_t^2\frac{\partial^2 H}{\partial y^2} \\
&\quad -\lambda_t(V-H)\left[\left(\frac{1}{\lambda_t}\right)' + \gamma\sigma_t^2\frac{\partial H}{\partial y}\right] \\
&\quad -c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[H^e(t,y+\lambda_t u)\,H(t,y+\lambda_t u) - H(t,y)\,H^e(t,y+\lambda_t u)\right. \\
&\quad \left.-u\lambda_t H^e(t,y)(t,y)\right]\nu_t(\mathrm{d}u). \qquad (20)
\end{aligned}
$$

We remark that the equation differs in two terms from the one in Cho [5]: the first term is given by the presence of the drift μ and the last term which is given because of the jumps. If there are no jumps and drift, a solution can be found as done in Cho [5].

Suppose that we have drift and diffusion part but that there are no jumps in the noise traders' process. The last equation reduces to

$$\begin{aligned} 0 &= -H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}+\frac{\partial H}{\partial t}+\frac{1}{2}\sigma_t^2\lambda_t^2\frac{\partial^2 H}{\partial y^2} \\ &\quad -\lambda_t(V-H)\left[\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}\right]. \end{aligned}$$

Then

$$H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}+\lambda_t(V-H)\left[\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}\right]$$

cannot depend on V, equivalently, by differentiating with respect to V, we have

$$H^e(t,y)\,\frac{\gamma}{\lambda_t}yc_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}=\lambda_t\left[\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}\right] \tag{21}$$

where, for $\mu_t\neq 0$, the right hand side is strictly increasing in V, while the the left hand side does not depend on it, which is a contradiction. Hence, we can have a solution only if $\mu_t\equiv 0$ which implies

$$\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}=0.$$

Note that this is the same situation as in Cho [5]. With analogous reasoning, one can show that, allowing jumps and drift only we arrive to a contradiction.

In fact the equation (20) has the form

$$\begin{aligned} 0 &= -H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}+\frac{\partial H}{\partial t}+ \\ &\quad -\lambda_t(V-H)\left(\frac{1}{\lambda_t}\right)' \\ &\quad -c_2(t)\,\frac{\gamma}{\lambda_t}\int_{\mathbb{R}}\left[H^e(t,y+\lambda_t u)\left(H(t,y+\lambda_t u)-H(t,y)\right)\right. \\ &\quad \left.-u\lambda_t H^e(t,y)\,\frac{\partial H}{\partial y}(t,y)\right]\nu_t(\mathrm{d}u), \end{aligned}$$

therefore,

$$\begin{aligned} &-H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}-\lambda_t(V-H)\left(\frac{1}{\lambda_t}\right)' \\ &-c_2(t)\,\frac{\gamma}{\lambda_t}\int_{\mathbb{R}}\left[H^e(t,y+\lambda_t u)\left(H(t,y+\lambda_t u)-H(t,y)\right)\right. \\ &\left.-u\lambda_t H^e(t,y)\,\frac{\partial H}{\partial y}(t,y)\right]\nu_t(\mathrm{d}u), \end{aligned}$$

does not depends on V. Then, by differentiation with respect to V, we obtain

$$\begin{aligned} 0 &= H^e(t,y)\,c_2(t)\,\frac{\gamma^2}{\lambda_t} y\mu_t \frac{\partial H}{\partial y} - \lambda_t \left(\frac{1}{\lambda_t}\right)' \\ &\quad + c_2(t)\,\frac{\gamma^2}{\lambda_t^2} \int_{\mathbb{R}} \left[(y+\lambda_t u)\,H^e(t,y+\lambda_t u)\,(H(t,y+\lambda_t u) - H(t,y))\right. \\ &\quad \left. - u\lambda_t y H^e(t,y)\,\frac{\partial H}{\partial y}(t,y)\right] \nu_t(\mathrm{d}u). \end{aligned}$$

or equivalently,

$$\begin{aligned} \frac{\lambda_t^2}{c_2(t)\,\gamma^2}\left(\frac{1}{\lambda_t}\right)' &= yH^e(t,y)\,\mu_t \frac{\partial H}{\partial y} \\ &\quad + \frac{1}{\lambda_t}\int_{\mathbb{R}} \left[(y+\lambda_t u)\,H^e(t,y+\lambda_t u)\,[H(t,y+\lambda_t u) - H(t,y)]\right. \\ &\quad \left. - u\lambda_t y H^e(t,y)\,\frac{\partial H}{\partial y}(t,y)\right] \nu_t(\mathrm{d}u). \end{aligned}$$

By differentiating again with respect to V, we obtain

$$\begin{aligned} 0 &= y^2 H^e(t,y)\,\mu_t \frac{\partial H}{\partial y} \\ &\quad + \frac{1}{\lambda_t}\int_{\mathbb{R}} \left[(y+\lambda_t u)^2 H^e(t,y+\lambda_t u)\,[H(t,y+\lambda_t u) - H(t,y)]\right. \\ &\quad \left. - u\lambda_t y^2 H^e(t,y)\,\frac{\partial H}{\partial y}(t,y)\right] \nu_t(\mathrm{d}u) \\ 0 &= y^2 \mu_t \frac{\partial H}{\partial y} \\ &\quad + \frac{1}{\lambda_t}\int_{\mathbb{R}} \left[(y+\lambda_t u)^2 H^E \exp\{-\gamma V u\}\,[H(t,y+\lambda_t u) - H(t,y)]\right. \\ &\quad \left. - u\lambda_t y^2 \frac{\partial H}{\partial y}(t,y)\right] \nu_t(\mathrm{d}u), \end{aligned}$$

where H^E denotes $\exp\left\{\frac{\gamma}{\lambda_t}\int_y^{y+\lambda_t u} H \mathrm{d}w\right\} > 0$. So again, we have an equation with the left hand side is independent of V, but the right hand side is strictly decreasing in V.

Note that we obtain the same results having only jumps, with the drift part being zero. So in the risk-averse case we can expect to find a solution to the existence of an equilibrium only in the case in which the noise trader's demand process presents only a diffusion part.

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